Superconductivity Within a Slab

exb2

Model of a superconducting slab in a parallel magnetic field.

Variables and parameters:

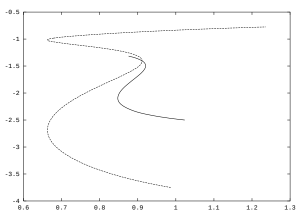
- λ: square of external field (bifurcation parameter)
- d: thickness of the slab ($d = 5$)
- x: space variable, $0 \le x \le d$
- κ: Ginzburg–Landau parameter ($\kappa = 1$)
- Θ: an order parameter characterizing different superconducting states
- Ψ: potential of the magnetic field

The Ginzburg–Landau equations are equivalent to the boundary-value problem of two second-order ODEs

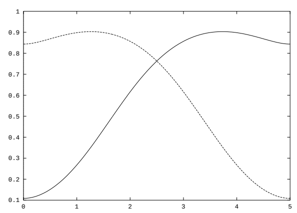
$$\Theta'' = \Theta(\Theta^2 - 1 + \lambda\Psi^2)\kappa^2$$
$$\Psi'' = \Theta^2\Psi$$

$$\Theta'(0) = \Theta'(d) = 0, \quad \Psi'(0) = \Psi'(d) = 1.$$

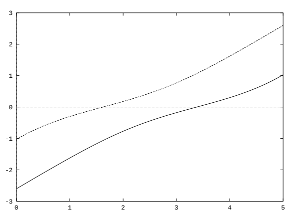
Figures:

1 

Bifurcation diagram $\Psi(0)$ versus λ. Two branches intersecting in a pitchfork bifurcation.

2 

Two antisymmetric solutions $\Theta(x)$ for $\lambda = 0.6647$.

3 

Two antisymmetric solutions $\Psi(x)$ for $\lambda = 0.6647$.

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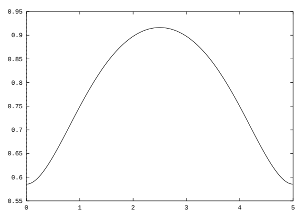
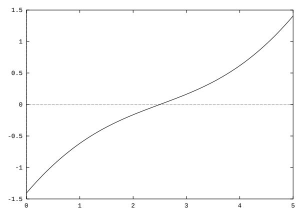


Symmetric solution $\Theta(x)$ for $\lambda = 0.91196$.

5



Symmetric solution $\Psi(x)$ for $\lambda = 0.91196$.

Comments and explanations:
Original source of the model: F. Odeh: Existence and bifurcation theorems for the Ginzburg-Landau equations. J. Math. Phys. **8** (1967) 2351–2356.
Solutions are summarized in the branching diagram in Figure 1. A branch of asymmetric solutions branches off a branch of symmetric solutions at a pitchfork bifurcation point. The data of the bifurcation point and four turning points (TP) are listed in the Table.
Investigation of the above boundary-value problem reveals that it supports symmetry. To see this, define

$$\tilde{\Theta}(x) := \Theta(d - x), \quad \tilde{\Psi}(x) := -\Psi(d - x)$$

and realize that the signs are the same after differentiating twice with respect to x. Hence, $\tilde{\Theta}$ and $\tilde{\Psi}$ satisfy the differential equations if Θ and Ψ do. Analyzing the boundary conditions, as, for example,

$$\tilde{\Theta}'(0) = -\Theta'(d) = 0,$$

shows that $\tilde{\Theta}$ and $\tilde{\Psi}$ satisfy the boundary conditions. Hence, the equation supports even Θ and odd Ψ. This does not imply that solutions must be even or odd. In fact, the equation has asymmetric solutions. See the solution in Figure 2, which depicts both the solutions Θ, Ψ and $\tilde{\Theta}, \tilde{\Psi}$ for the parameter value $\lambda = 0.6647$.

In order to calculate solutions, we rewrite the boundary-value problem as a first-order system. Denoting $y_1 = \Theta$, $y_2 = \Theta'$, $y_3 = \Psi$, and $y_4 = \Psi'$ yields

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1(y_1^2 - 1 + \lambda y_3^2) \\ y_3' &= y_4 \\ y_4' &= y_1^2 y_3 \end{aligned}$$

$$y_2(0) = y_2(5) = 0, \quad y_4(0) = y_4(5) = 1$$

The solution profiles of the bifurcation point are shown in Figures 4 and 5; notice the symmetries. This bifurcation point is straddled between the two turning points with asymmetric solutions, which are depicted in Figures 2 and 3. Compare the figures to see how this straddling becomes evident in the solution profiles.

TABLE: bifurcation points

λ_0	$\Theta_0(0)$	$\Psi_0(0)$	Type
0.66294	0.08996	-2.687	TP, asymmetric sol.
0.66294	0.8461	-1.016	TP, asymmetric sol.
0.91196	0.5853	-1.407	Pitchfork bif. point
0.92070	0.5214	-1.496	TP, symmetric sol.
0.84769	0.1923	-2.095	TP, symmetric sol.