Forced Duffing Oscillator

exb3

Variable and parameter:

u: state variable of an oscillator

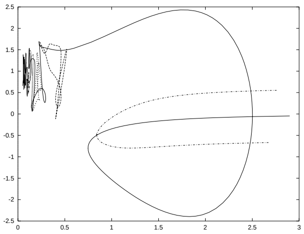
ω: frequency of the driving force (bifurcation parameter)

differential equation:

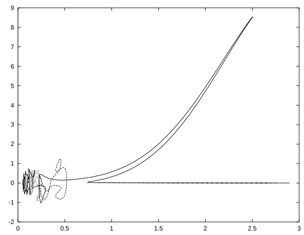
$$\ddot{u} + \frac{1}{25}\dot{u} - \frac{1}{5}u + \frac{8}{15}u^3 = \frac{2}{5}\cos\omega t.$$

We basically look for *harmonic* solutions, which are periodic with period $T = 2\pi/\omega$.

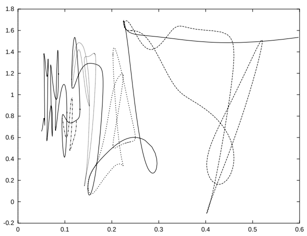
Figures:

1 

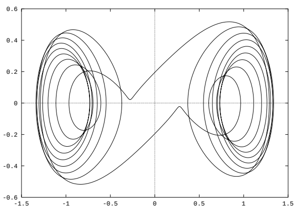
Bifurcation diagram $u(0)$ versus ω. Solid curve: branch of symmetric orbits.

2 

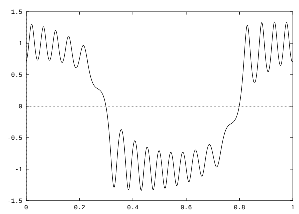
Bifurcation diagram $\dot{u}(0)$ versus ω.

3 

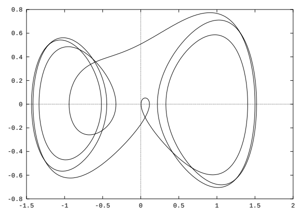
Detail of bifurcation diagram, $u(0)$ versus ω.

4 

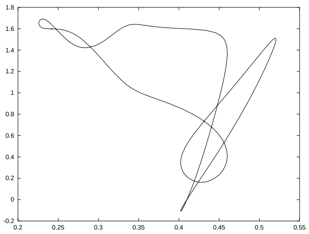
$(u, \dot{u})$ phase-plane for $\omega = 0.053$, symmetric oscillation.

5 

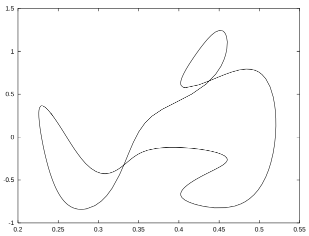
$u(t)$ for $\omega = 0.053$, t is a normalized time, $t = 1$ corresponds to $T = 2\pi/\omega = 118.55$.

6 

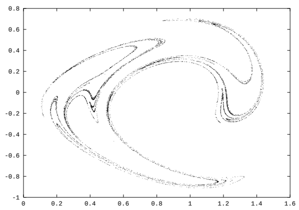
$(u, \dot{u})$ plane for $\omega = 0.12861$, asymmetric oscillation.

7 

Bifurcation diagram of a closed branch of asymmetric solutions, $u(0)$ versus ω.

8 

Same as in Figure 7, but $\dot{u}(0)$ versus ω.

9 

Duffing equation, $(u, \dot{u})$ plane, Poincaré set of the stroboscopic map, $\omega = 0.2$, fractal dimension $D = 2.381$.

Comments and explanations:

Derivation of the equation:

Duffing equations are used to model certain mechanical or electrical oscillators. One frequently mentioned derivation as a one-mode approximation of the deflections $v(x,t) = u(t)\sin \pi x$ of a symmetric beam modelled by

$$v_{xxxx} + \left(\Gamma - \mathrm{K}\int_0^1 (v_x(\xi,t))^2 d\xi\right) v_{xx} + \delta v_t + v_{tt} = P$$

with

$$P(x,t) = \gamma \cos \omega t \sin \pi x, \quad 0 \le x \le 1,$$
$$\Gamma = \pi^2 + 0.2\pi^{-2}, \quad K = 16\pi^{-4}/15,$$
$$\gamma = 0.4, \quad \delta = 0.04$$

is summarized in [R. Seydel: Practical Bifurcation and Stability Analysis. From Equilibrium to Chaos. Second Edition. Springer Interdisciplinary Applied Mathematics 1994.]

It is an instructive exercise to consider the case $\gamma = 0$ (no driving force) with the compressive force Γ as bifurcation parameter. For $\Gamma = \pi^2$ there is a pitchfork bifurcation. At this value of *Euler's first buckling load* the underlying beam starts to bend. Up to symmetry, one stable equilibrium exists for each $\Gamma > \pi^2$. This is satisfied by our parameters.

First-order System:

The harmonic forcing term $0.4 \cos \omega t$ on the right-hand side provokes a response of the system (beam, electric current). The harmonic oscillations obey the boundary conditions

$$u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T).$$

Because the period T varies with ω, the integration interval $0 \le t \le T$ must be adapted whenever ω is changed. This is inconvenient when nodes of a numerical solution procedure must also be adapted, so we transform the integration interval to unit length, thereby shifting the dependence on ω to the right-hand side of the differential equation. The normalized time $\tilde{t}$ satisfies

$$T\tilde{t} = t, \quad 0 \le \tilde{t} \le 1.$$

The transformation

$$y_1(\tilde{t}) := u(t), \quad y_2(\tilde{t}) := \dot{u}(t)$$

leads to the first-order system

$$\begin{aligned} y_1' &= Ty_2, \\ y_2' &= T(-0.04y_2 + 0.2y_1 - 8y_1^3/15 + 0.4\cos 2\pi\tilde{t}) \end{aligned}$$

with boundary conditions

$$\begin{aligned} y_1(0) - y_1(1) &= 0, \\ y_2(0) - y_2(1) &= 0. \end{aligned}$$

Redefining the normalized time by t, we obtain a boundary-value problem of a standard form. With $y_3 := \omega$, the corresponding extended system is

$$\begin{aligned} y_1' &= 2\pi y_2/y_3, \\ y_2' &= 2\pi(-0.04y_2 + 0.2y_1 - 8y_1^3/15 + 0.4\cos 2\pi t)/y_3, \\ y_3' &= 0, \end{aligned}$$

$$\begin{aligned} y_1(0) - y_1(1) &= 0, \\ y_2(0) - y_2(1) &= 0, \\ y_k(0) - \eta &= 0. \end{aligned}$$

Bifurcations:

A main part of the bifurcation diagram is shown in Figures 1, 2. The branching behavior is rich for small values of ω. As Figure 3 shows, there are many branches, turning points, and bifurcation points for values of the parameter $\omega < 0.5$. The solid curve in this branching diagram represents oscillations, with phase diagrams being symmetric with respect to the origin as in Figure 4. For decreasing ω, each loop of this "main" branch attaches a further wiggle to the oscillation. The wiggles indicate that the small-amplitude oscillation around any of the stable equilibria takes much time to collect enough energy before a transition to the other attracting basin is possible; compare the phase diagram in Figure 4 and the time dependence in Figure 5 ($\omega = 0.053$). At first glance, the many wiggles might give the impression that such solutions are not harmonic; but they do have period T. Each of the closed branches in Figure 3 (dashed and dotted curves) is attached to the main branch via two pitchfork bifurcations. Accordingly, the phase diagrams of these "secondary" branches are asymmetric with respect to the origin; Figure 6 shows one such phase plot. The closed branch in Figure 7 (drawn in Figure 3 in a dashed line) appears to have sharp corners, and one might expect difficulties in tracing this particular branch. The plotted behavior refers only to the dependence $y_1(0)$ versus λ; the corresponding graph of $y_2(0)$ behaves more smoothly (Figure 8).

The table includes 20 bifurcation points of the above Duffing equation with accurate values of the amplitude. The above Duffing equation also exhibits *subharmonic* oscillations; then the period of the system is an integral multiple of the period of the external frequency. Branches of subharmonic solutions bifurcate from the harmonic solutions (not listed in the table). Stability, and period doubling scenarios have been calculated by K. Riedel (Ulm 1996). For example, a period doubling cascade starts at 0.88846 ($u(0) = -0.66$).

TABLE: 20 bifurcation points of the Duffing equation

ω	amplitude	type of bifurcation
2.502052	4.110751	turning point
0.842728	0.494041	pitchfork bifurcation
0.748053	0.798077	turning point
0.520572	1.509612	pitchfork bifurcation
0.460134	1.627274	turning point
0.402191	1.669646	turning point
0.296267	1.538703	turning point
0.250065	1.385417	pitchfork bifurcation
0.225884	1.673456	pitchfork bifurcation
0.224722	1.684075	turning point
0.224279	1.487556	turning point
0.202391	1.596173	turning point
0.181368	1.456409	turning point
0.165875	1.369795	pitchfork bifurcation
0.149098	1.600829	pitchfork bifurcation
0.148825	1.604388	turning point
0.132017	1.422928	turning point
0.124410	1.355929	pitchfork bifurcation
0.114505	1.542689	pitchfork bifurcation
0.114355	1.544986	turning point

Chaotic solutions:

For a range of ω-values ther exist chaotic attractors. One such attractor is illustrated by the Poincaré map / stroboscopic map $(u(t_k), \dot{u}(t_k))$, $t_k = \frac{2\pi}{\omega}$, $k = 1, 2, ..., 8000$, $\omega = 0.2$, $u(0) = \dot{u}(0) = 0$ in Figure 9.