to appear in: Int. J. Bifurcation and Chaos

A Model of Student Migration

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Recently, along with diminishing numbers of students, it has been observed that some graduate programs still have lots of students, whereas other fields of study suffer from a great loss of students. Apparently, students do not distribute evenly over the fields a university offers. The relations among the number of students in different fields of concentration are not constant. In particular, the harder subjects appear to suffer more from a diminishing number of students than the fields with a reputation of being somewhat "softer."

The reasons for such trends must be expected to be related to the size of the market, and to psychological effects. Our study has been motivated by attempts to model economic mechanisms that are related to advertising [1]. We have constructed a mathematical model that is able to explain trends such as reported above.

1. Choice Based on Communication

Occasionally one faces the necessity to choose between a popular option with little risk on the one side, and a demanding and more risky option on the other side. Such an alternative is a standard situation for students at universities. For example, the option 1 may be to major in a subject with the reputation of comparably easy examinations with favorable grades, and with comfortable workload. The option 2 may be a field or a professor having the reputation of being more exhaustive, or more ambitious. According to these estimates, option 1 will probably be a frequent choice, and option 2 will be chosen more seldomly.

What is the mechansim that leads students to choose option 2 anyway? Typically, option 2 is chosen only when the student's fears are appeased by the fact that other students manage well with option 2 and do some advertising for option 2 by communication among their fellow students. When by good word of mouth the information is that the professor who represents option 2 is friendly and the level is not too difficult, then the advertising for option 2 is by communication.

2. The Model

To set up a model, we divide the number of students of a particular field into $X_1(t)$ and $X_2(t)$. We denote by $X_1(t)$ the number of individuals who follow option 1 at time instant t, and $X_2(t)$ is the number of those individuals who are following option 2. We assume that the migration from X_1 to X_2 is dominated by communication, as discussed in Section 1. We denote ΔX_i the change in X_i within a time interval Δt, $i = 1, 2$. Then the migration within Δt from the X_1-group to the X_2-group of individuals is cX_1, where c symbolizes the effect of communication — that is, $c = a(t)X_2(t)$. According to this model, the transition from X_1 to X_2 is given by the nonlinear term $a(t)X_1(t)X_2(t)$.

The factor $a(t)$ measures in particular the contact rate, and the effectiveness of the communication.

The remaining parts of the model are the input λ of fresh students per time interval Δt, which is supposed to be constant, and the rates d_1 and d_2 of successful final examinations. Finally we have to compensate for those students who are disappointed by X_2, and go back to the mainstream X_1. The rate of this backflow is denoted β. In summary, the migrations are given by

$$\begin{aligned} \Delta X_1 &= X_1(t_{j+1}) - X_1(t_j) = \lambda - a(t_j)X_1(t_j)X_2(t_j) + \beta X_2(t_j) - d_1 X_1(t_j) \\ \Delta X_2 &= X_2(t_{j+1}) - X_2(t_j) = a(t_j)X_1(t_j)X_2(t_j) - \beta X_2(t_j) - d_2 X_2(t_j). \end{aligned} \tag{1}$$

Since students enter the university at discrete time instances, we discretize time to t_j, $j = 0, 1, 2, \ldots$. For example, Δt may be the time period of one semester, and t_j is the end of the jth semester and the beginning of the $(j+1)$st semester, respectively. Freshmen can only enter into the X_1-group because the choice between X_1 and X_2 is thought as being a specialization later during the studies, say, at the graduate level. The iteration based on (1) is $X_i(t_{j+1}) = X_i(t_j) + \Delta X_i(t_j), \; i = 1, 2$.

Some restrictions on X_1, X_2 are natural. First we require

$$X_1(t) \geq 0, \; X_2(t) \geq 0. \tag{2}$$

Furthermore, we must be sure that only graduate students may specialize into X_2. Let $\gamma X_1(t)$ be those students who have finished with the undergraduate school, and who are allowed to enter X_2. Then we have the bound

$$a(t)X_1(t)X_2(t) \leq \gamma X_1(t) \tag{3}$$

on the migration towards X_2. Figure 1 summarizes the model. We note in passing that one can dispense with the bound (3) when X_1 is interpreted as the graduate portion of the mainstream students; the results of this modification are analogous. We shall say a bound is *active* if "=" holds.

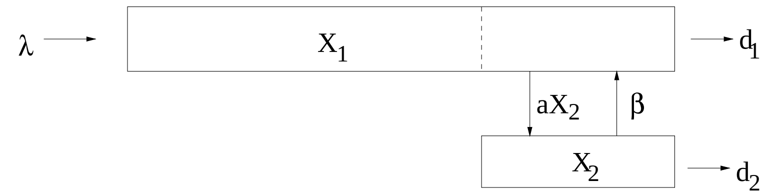

Figure 1: Schematic view of the model

3. Technical Assumptions

We introduce the "fractional student," $X_i \in \mathbb{R}$, for an easier discussion of convergence. Further we assume $a(t_j) \geq 0$, and a maximum value $a_{\max}$ with $a(t_j) \leq a_{\max}$. We take $a(t_j) = a_{\max}$ as long as the bounds (2) and (3) are not active. To take reasonable numbers for numerical experiments we take a semester szenario with 10 semesters as average time to complete the studies, and assume about 3 semesters to complete the X_2-option successfully. The portion of graduate students is assumed 40%. The constant β is chosen artificially to be $\beta = 0.1$. This szenario is represented by

$$d_1 = 0.1, \; d_2 = 0.3, \; \beta = 0.1, \; \gamma = 0.4. \tag{4}$$

The remaining parameters are λ and $a_{\max}$. Here we take several choices. Note that these numbers are for illustration only; the theoretical results (Section 5) are independent of the choice of the numbers.

4. Some Experiments

Denote $y_1^j := X_1(t_j)$, $y_2^j := X_2(t_j)$. We illustrate some experiments run with the chosen numbers. First we start from $X_1(t_0) = 200$, $X_2(t_0) = 20$ and iterate (1) over 40 semesters, for $\lambda = 50$, $a = 0.002$. The time history is shown in Figure 2; Figure 3 depicts the corresponding phase diagram. As can be seen from these figures, the number of students drastically changes during the first few semesters, and then quickly settles into a *stationary* situation with constant X_1, X_2. For the corresponding value of $a = 0.002$ the bounds (2), (3) are not active. Increasing a, the bound (3) becomes important.

It is most interesting to investigate the stationary situation for various values of a, λ. These numbers are changed quasistationary. That is, each pair $(a_{\max}, \lambda)$ is kept constant for each simulation, but each simulation works with a different pair. In Table 1 we illustrate the dependence on a for $\lambda = 50$. The final entries show that a remains on a constant level after $a_{\max}$ passes the threshold where the bound (3) becomes active. In Table 2 the parameter λ is varied. For the underlying value of $a = 0.001$, $X_2 \equiv 0$ for $\lambda \leq 40$. This looks like a bifurcation with an exchange of stability. In fact, this is easily proved in Section 5.

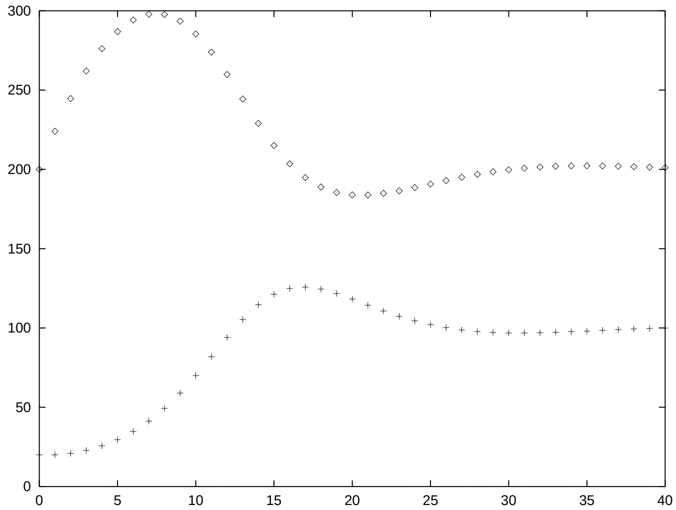

Figure 2: Iteration (1), $y_1(t_j)$ (rhombuses), $y_2(t_j)$ (crosses) for $j = 0, 1, ..., 40$, $\lambda = 50$, $a = 0.002$.

$a_{\max}$	y_1^∞	y_2^∞	a^∞
0.0005	500	0	0.0005
0.0007	500	0	0.0007
0.0009	444.4	18.5	0.0009
0.0030	133.3	122.2	0.0030
0.0035	125	125	0.0032
0.0100	125	125	0.0032

Table 1: Fixed points of the equation (1) depending on a, for $\lambda = 50$.

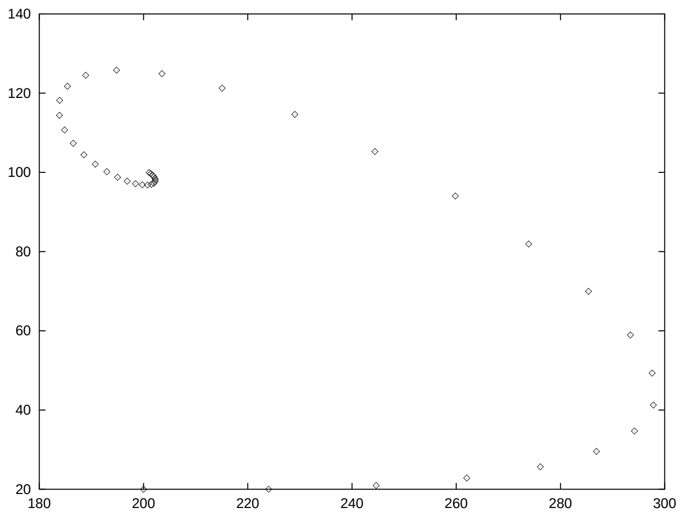

Figure 3: Iteration (1), phase-diagram of the (y_1^j, y_2^j)–values for $j = 0, 1, ..., 40$ and $\lambda = 50$, $a = 0.002$.

λ	y_1^∞	y_2^∞
20	200	0
30	300	0
40	400	0
41	400	3.333
45	400	16.666
50	400	33.333
100	400	200
200	500	500

Table 2: Dependence of the fixed points of (1) on the parameter λ, for $a = 0.001$ except for $\lambda = 200$ where (3) is active.

5. Bifurcation of the Fixed Points

The fixed points can be calculated by an easy hand calculation. One fixed point is

$$X_1 = \frac{\lambda}{d_1},\ X_2 = 0; \tag{5}$$

the other fixed point is

$$X_1 = \frac{\beta + d_2}{a},\ X_2 = \frac{1}{d_2}\left(\lambda - \frac{d_1}{a}(\beta + d_2)\right). \tag{6}$$

Since $X_2 \geq 0$ must hold, there is a bifurcation for those parameter combinations for which $X_2 = 0$. This leads to the relation

$$\lambda a = d_1(\beta + d_2), \tag{7}$$

which defines the bifurcation.

A standard stability analysis [2] based on the Jacobian

$$J = \begin{pmatrix} -aX_2 - d_1 & -aX_1 + \beta \\ aX_2 & aX_1 - \beta - d_2 \end{pmatrix}$$

for the two fixed points shows that the fixed point (5) is stable for $a\lambda < d_1(\beta + d_2)$, and the fixed point (6) is stable for $X_2 > 0$, or $a\lambda > d_1(\beta + d_2)$. This analysis does not take into consideration the bound (3), which is clearly satisfied for fixed point (5). In summary, we have the result:

Lemma: For $a\lambda < d_1(\beta + d_2)$ the stationary solution (5) with $X_2 = 0$ is stable.

In case the bound (3) is active, $aX_1X_2 = \gamma X_1$ can be substituted into equation (1). Then the stationary solution of the corresponding modified iteration is obtained as

$$\begin{aligned} X_1 &= \lambda \frac{\beta + d_2}{\beta d_1 + \gamma d_2 + d_1 d_2} \\ X_2 &= \lambda \frac{\gamma}{\beta d_1 + \gamma d_2 + d_1 d_2}, \end{aligned} \tag{8}$$

where $d_1X_1 + d_2X_2 = \lambda$ holds. This solution is independent of a. By further manipulations on can calculate a relation between a and λ such that the "free" branch of equation (6) meets the "boundary" solution (8). The criterion is

$$a\lambda = \beta d_1 + \gamma d_2 + d_1 d_2. \tag{9}$$

This limits the free branch (6) to the domain

$$d_1(\beta + d_2) < a\lambda < d_1(\beta + d_2) + \gamma d_2. \tag{10}$$

For the choice of constants (4), and $a = 0.001$, this yields the interval

$$40 < \lambda < 160,$$

and explains the results of Table 2. In summary, the bifurcation chart of the fixed points is illustrated by Figure 4. The subdomains in Figure 4 are indicated by the number of the equation defining the proper fixed point.

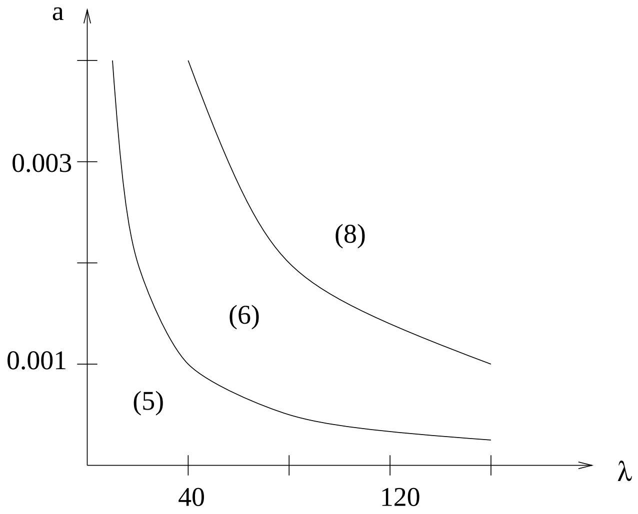

Figure 4: Bifurcation chart of fixed points for the choice of constants (4)

6. Discussion

Since the representatives of option 2 face the serious situation to have no students in case

$$\lambda < d_1 \frac{\beta + d_2}{a},$$

they can improve their situation by taking care that this inequality does not become true. This leads to the strategies

(a) advertising towards more freshmen entering X_1 (increase λ), or
(b) improving the communication (increase a), or
(c) make the X_2-students feel happier (decrease β), or
(d) help them to finish quicker (decrease d_2).

These stategies are derived from our model, but are apparently obvious. What is remarkable and unexpected is the bifurcation. This fact points at the existence of a threshold value of the number λ of entering freshmen. If λ falls below the threshold, option 2 has no chance to have students, according to our model. We also see that it does not make sense to increase the advertising (a_{max}) beyond a certain level.

Computer demonstration. A computer demonstration can be downloaded from the domain *www.bifurcation.de.*

Acknowledgement. This work was sponsered by Volkswagenstiftung (RiP-program, Oberwolfach).

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